

q-Statistical mechanics of phase-space curves

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April 14, 2017

Abstract

We study the non additive, classical q-statistical mechanics of a phase-space curve. This allows one to disclose an entropic force mechanism that yields a simple realization of interesting effects, such as confinement, hard core, and asymptotic freedom.

Keywords: Phase-space curves; Entropic force; q-are statistical mechanic

1 Introduction

1.1 q-Statistics

The so called q-statistical mechanics has been used in multiple applications in the last years [1, 2, 3, 4, 5, 6, 7, 8, 9, 10, 11, 12], being of great relevance for astrophysics, particularly for self-gravitating systems [13, 14, 15]. Additionally, it was useful in variegated scientific fields. It has originated several thousands of both papers and authors [2]. Research on its structural features is important for astronomy, physics, neurology, biology, economic sciences, etc. Its success reaffirms the idea that a great deal of physics derives from exclusively statistical considerations, rather than mechanical ones. A foremost example lies in its application to high energy physics. Here q-statistics seems to adequately describe the transverse momentum distributions of different hadrons [16, 17, 18]. In this sense, $q = 1.15$ has acquired particular relevance [16, 17, 18].

1.2 Our goals

One investigates in this work the classical q-statistical mechanics of arbitrary phase-space curves Γ , disclosing in such a way interesting features, like confinement and hard-cores. 1) Of course, by confinement we are reminded of the phenomenon that impedes isolation of color charged particles (quarks that cannot be isolated singularly) and thus cannot be directly detected. 2) By asymptotic freedom we make reference to a property of some gauge theories that generates particles' bonds to become asymptotically weaker as distance diminishes. 3) In a "hard core" repulsive scenario each particle (a nucleon, for instance) consists of a hard core endowed with an infinitely repulsive potential. Our curves-research will yield a simple, classical entropic mechanism for the above cited phenomena.

Remind that the entropic force is a phenomenological one that derives from statistical tendencies to entropic growth [19, 20, 21, 22, 23, 24, 25], without appealing to any specific underlying microscopic interaction. The foremost example is the elasticity of a freely-jointed polymer molecules [19, 20]. It is remarkable that Verlinde has argued that gravity can also be understood in terms of an entropic force [21]. Ditto for the Coulomb force [26], etc. [27].

Here we devise a very simple q-model to demonstrate that confinement can

emerge from entropic forces, by appeal to a quadratic Hamiltonian in phase-space. These Hamiltonians are well known, classically and quantumly. For them, the correspondence between classical and quantum mechanics is exact. Unfortunately, explicit formulas are not trivial.

Knowledge of quadratic Hamiltonians is of utility for investigating more general Hamiltonians (and their associated Schroedinger equations) in a semi-classical scenario. Quadratic Hamiltonians are relevant in partial differential equations: they yield non trivial instances of wave propagation phenomena. Quadratic Hamiltonians also help in gaining insight into properties of more involved Hamiltonians used in quantum theory.

We thus will appeal to quadratic Hamiltonians in a classical environment so as to learn whether some interesting properties emerge concerning the entropic force along phase-space curves, which will indeed be the fact. A similar previous analysis involving the $q = 1$, ordinary Boltzmann entropy has been reported in [24].

2 Preliminaries

We consider a harmonic oscillator-like Hamiltonian

$$H(P, Q) = P^2 + Q^2, \quad (2.1)$$

where both P^2 and Q^2 have the dimension of H . The partition function in Tsallis statistical is defined as [1]

$$Z(\beta) = \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} [1 + (q - 1)\beta H(P, Q)]^{\frac{1}{1-q}} dP dQ, \quad (2.2)$$

where $1 \leq q < 2$ is the non-extensibility parameter. Note that, if $q \rightarrow 1$, the partition function reduces to the usual one, that means, the Gibbs-Boltzmann's partition function. Appealing to the change of variables

$$U = P^2 + Q^2, \quad (2.3)$$

we obtain

$$Z(\beta) = \pi \int_0^{\infty} [1 + (q - 1)\beta U]^{\frac{1}{1-q}} dU. \quad (2.4)$$

Evaluating now (2.4) we have

$$Z(\beta) = \frac{\pi}{\beta(q-1)} B \left[1, \frac{2-q}{q-1} \right], \quad (2.5)$$

where $B[a, b]$ is the beta function. Thus,

$$Z(\beta) = \frac{\pi}{\beta(q-1)} \frac{\Gamma(1)\Gamma\left(\frac{2-q}{q-1}\right)}{\Gamma\left(\frac{1}{q-1}\right)}, \quad (2.6)$$

or equivalently,

$$Z(\beta) = \frac{\pi}{\beta(2-q)}. \quad (2.7)$$

If $q \rightarrow 1$, then

$$Z \rightarrow \frac{\pi}{\beta}, \quad (2.8)$$

and Eq. 2.7 thus reduces to the expression obtained of [24] for the Gibbs-Boltzmann statistics.

Similarly, for the mean value of the energy we have

$$\langle u \rangle (\beta) = \frac{1}{Z} \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} H(P, Q) [1 + (q-1)\beta H(P, Q)]^{\frac{1}{1-q}} dP dQ, \quad (2.9)$$

or

$$\langle u \rangle (\beta) = \frac{\pi}{Z} \int_0^{\infty} u [1 + (q-1)\beta u]^{\frac{1}{1-q}} du. \quad (2.10)$$

The result of (2.10) is

$$\langle u \rangle (\beta) = \frac{\pi}{\beta^2(q-1)^2 Z} B \left[2, \frac{3-2q}{q-1} \right], \quad (2.11)$$

that can be recast as

$$\langle u \rangle (\beta) = \frac{\pi}{\beta^2 Z} \frac{1}{(2-q)(3-2q)}. \quad (2.12)$$

Replacing above the value of Z we obtain

$$\langle u \rangle (\beta) = \frac{1}{\beta(3-2q)}, \quad (2.13)$$

with the restriction $1 \leq q < 1.5$ in order to guarantee the non-divergence of $\langle U \rangle$.

When $q \rightarrow 1$ we obtain

$$\langle U \rangle \rightarrow \frac{1}{\beta}. \quad (2.14)$$

In [25] we found for the entropy

$$S(\beta) = \ln_{2-q} Z + Z^{q-1} \beta \langle U \rangle, \quad (2.15)$$

where $\ln_q(z)$ is the q -logarithm function defined as

$$\ln_q(z) = \frac{z^{1-q} - 1}{1 - q}. \quad (2.16)$$

Replacing things in (2.15) we get

$$S(\beta) = Z^{q-1} \left(\beta \langle U \rangle + \frac{1}{q-1} \right) - \frac{1}{q-1}, \quad (2.17)$$

and

$$S(\beta) = \left[\frac{\pi}{\beta(2-q)} \right]^{q-1} \frac{(2-q)}{(3-2q)(q-1)} - \frac{1}{q-1}. \quad (2.18)$$

3 Path entropy

We focus now on the concept of path entropy [24]. The path is a phase-space curve Γ parameterized by the variable Q . Following the procedure of [24], we shall specialize equations 2.4 and 2.10 to curves Γ . We first define

$$Z(\beta, \Gamma) = \pi \int_{\Gamma} [1 + (q-1)\beta U(P, Q)]^{\frac{1}{1-q}} dU(P, Q). \quad (3.1)$$

If we consider curves (parametrized by the independent variable Q) passing through the origin, we have $P(0) = 0$ and $Q = 0$, and consequently $U(0, 0) = 0$. Since the integrand in equation 3.1 is an exact differential and the integral only depends on the end point Q_0 we have

$$Z(\beta, Q_0) = \pi \int_0^{Q_0} [1 + (q-1)\beta U(P, Q)]^{\frac{1}{1-q}} dU(P, Q), \quad (3.2)$$

or

$$Z(\beta, Q_0) = \frac{\pi}{(2-q)\beta} \{1 - [1 + (q-1)\beta U(P(Q_0), Q_0)]^{\frac{2-q}{1-q}}\}. \quad (3.3)$$

If $Q_0 \rightarrow \infty$ then $Z(\beta, Q_0)$ reduces to expression 2.7. Moreover if $q \rightarrow 1$, then

$$Z(\beta, Q_0) \rightarrow \frac{\pi}{\beta} (1 - e^{-\beta U(P(Q_0), Q_0)}). \quad (3.4)$$

In the same way, we have for the mean value of energy

$$\langle U \rangle (\beta, \Gamma) = \frac{\pi}{Z(\beta, \Gamma)} \int_{\Gamma} U(P, Q) [1 + (q-1)\beta U(P, Q)]^{\frac{1}{1-q}} dU(P, Q), \quad (3.5)$$

or equivalently,

$$\langle U \rangle (\beta, Q_0) = \frac{\pi}{Z(\beta, Q_0)} \int_0^{Q_0} U(P, Q) [1 + (q-1)\beta U(P, Q)]^{\frac{1}{1-q}} dU(P, Q) \quad (3.6)$$

After to evaluate (3.6) we obtain:

$$\langle U \rangle (\beta, Q_0) = \frac{\pi}{Z(\beta, Q_0)\beta^2} \left\{ \frac{1 - [1 + (q-1)\beta U(P(Q_0), Q_0)]^{\frac{3-2q}{1-q}}}{(3-2q)(q-1)} - \frac{1 - [1 + (q-1)\beta U(P(Q_0), Q_0)]^{\frac{2-q}{1-q}}}{(2-q)(1-q)} \right\} \quad (3.7)$$

and, simplifying the last expression,

$$\langle U \rangle (\beta, Q_0) = \frac{1}{\beta(q-1)} \left\{ -1 + \frac{(2-q)}{(3-2q)} \frac{\{1 - [1 + (q-1)\beta U(P(Q_0), Q_0)]^{\frac{3-2q}{1-q}}\}}{\{1 - [1 + (q-1)\beta U(P(Q_0), Q_0)]^{\frac{2-q}{1-q}}\}} \right\}. \quad (3.8)$$

If $Q_0 \rightarrow \infty$, then $\langle U \rangle (\beta, Q_0)$ reduces to expression (2.13). If $q \rightarrow 1$, then

$$\langle U \rangle (\beta, Q_0) \rightarrow \frac{1 - (1 + \beta U)e^{-\beta U}}{(1 - e^{-\beta U})\beta}, \quad (3.9)$$

which is consistent with the results obtained in [24]. Again, we obtain the entropy via the partition function and the mean value of energy, i.e.,

$$S(\beta, Q_0) = \ln_{2-q} Z(\beta, Q_0) + Z(\beta, Q_0)^{q-1} \beta \langle U \rangle (\beta, Q_0), \quad (3.10)$$

or, equivalently,

$$S(\beta, Q_0) = \frac{1}{q-1} \left\{ \frac{(2-q)}{(3-2q)} \left[\frac{\pi}{(2-q)\beta} \right]^{(q-1)} \{1 - [1 + (q-1)\beta U(P(Q_0), Q_0)]^{\frac{2-q}{1-q}}\}^{(q-2)} \right. \\ \left. \{1 - [1 + (q-1)\beta U(P(Q_0), Q_0)]^{\frac{3-2q}{1-q}}\} - 1 \right\}. \quad (3.11)$$

4 Equipartition

We find that

$$\langle Q^2 \rangle = \frac{1}{Z} \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} Q^2 [1 + (q-1)\beta(P^2 + Q^2)]^{\frac{1}{1-q}} dP dQ, \quad (4.1)$$

or

$$\langle Q^2 \rangle = \frac{\pi}{2Z} \int_0^{\infty} U [1 + (q-1)\beta U]^{\frac{1}{1-q}} dU, \quad (4.2)$$

while along the curve Γ ,

$$\langle Q^2 \rangle(\beta, \Gamma) = \frac{\pi}{2Z(\beta, \Gamma)} \int_{\Gamma} U [1 + (q-1)\beta U]^{\frac{1}{1-q}} dU. \quad (4.3)$$

Thus,

$$\langle Q^2 \rangle(\beta, Q_0) = \frac{\pi}{2Z(\beta, Q_0)} \int_0^{Q_0} U [1 + (q-1)\beta U]^{\frac{1}{1-q}} dU. \quad (4.4)$$

This integral yields

$$\langle Q^2 \rangle(\beta, Q_0) = \frac{\pi}{2Z(\beta, Q_0)\beta^2} \left\{ \frac{1 - [1 + (q-1)\beta U(P(Q_0), Q_0)]^{\frac{3-2q}{1-q}}}{(3-2q)(q-1)} - \frac{1 - [1 + (q-1)\beta U(P(Q_0), Q_0)]^{\frac{2-q}{1-q}}}{(2-q)(1-q)} \right\}, \quad (4.5)$$

and then

$$\langle Q^2 \rangle(\beta, Q_0) = \frac{\langle U \rangle(\beta, Q_0)}{2}, \quad (4.6)$$

that is,

$$\langle Q^2 \rangle(\beta, Q_0) = \langle P^2 \rangle(\beta, Q_0) = \frac{\langle U \rangle(\beta, Q_0)}{2}, \quad (4.7)$$

which, for $Q_0 \rightarrow \infty$ gives

$$\langle Q^2 \rangle = \langle P^2 \rangle = \frac{\langle U \rangle}{2} = \frac{1}{2(3-2q)\beta}, \quad (4.8)$$

the q-equipartition recipe. It set an upper bound for q , $q < 3/2$.

5 Adiabatic paths

Consider now isotropic paths with $S = \text{constant}$. From equation (3.11) we see that the condition for $S = \text{constant}$ reads

$$\beta = C_1 \quad ; \quad U(P(Q_0), Q_0) = C_2, \quad (5.1)$$

$\forall Q_0$. Thus, for the curve $P = f(Q)$ and for our Hamiltonian, one should have

$$P^2 + Q^2 = (P + \delta P)^2 + (Q + \delta Q)^2, \quad (5.2)$$

or,

$$P\delta P = -Q\delta Q, \quad (5.3)$$

At the curve $P = f(Q)$ we confront $P\delta P = Pf'(Q)\delta Q$ and

$$f(Q)f'(Q) = -Q, \quad (5.4)$$

turns out to be the condition for an adiabatic path $f(Q)$. We deal actually with an infinite family of paths, on account of the integration constant C here involved. The solution to (5.4) appeals first to a transformation into

$$\frac{df^2}{dQ} = -2Q, \quad (5.5)$$

$$f^2(Q) = -Q^2 + C \rightarrow P^2 + Q^2 = C. \quad (5.6)$$

We ask now for two straight-line paths, both i) passing through the origin and ii) having the same entropy. One faces

$$P(Q) = \alpha Q, \quad (5.7)$$

and

$$U = (\alpha^2 + 1)Q_0^2 = (\alpha'^2 + 1)Q_0'^2. \quad (5.8)$$

Should we have

$$\alpha' < \alpha, \quad (5.9)$$

and

$$Q'_0 = \sqrt{\frac{\alpha^2 + 1}{\alpha'^2 + 1}} Q_0, \quad (5.10)$$

one deduces that

$$\Delta S = S(\beta, \alpha', Q'_0) - S(\beta, \alpha, Q_0) = 0. \quad (5.11)$$

For a system 1) evolving from the line $\mathbf{p} = \alpha Q$, 2) ending up in the line $\mathbf{p} = \alpha' Q'$, and 3) traversing all the space between both lines, if (5.1) is complied with, we have adiabatic evolution.

6 Entropic force

According to [21] the entropic force F_e is given by

$$F_e = \frac{1}{\beta} \frac{\partial S}{\partial Q}. \quad (6.1)$$

In our case that is

$$F_e = \frac{Z^{q-2}}{\beta} \frac{\partial Z}{\partial Q} + (q-1) Z^{q-2} \frac{\partial Z}{\partial Q} \langle \mathbf{U} \rangle + Z^{q-1} \frac{\partial \langle \mathbf{U} \rangle}{\partial Q}. \quad (6.2)$$

Since Q_0 is arbitrary, we replace it for Q . Z and $\langle \mathbf{U} \rangle$ were determined in 3.3 and 3.8, respectively. Here from we simply speak of use denote Z and $\langle \mathbf{U} \rangle$ so as to simplify the notation, but we have to remind that these quantities are functions of β and Q . In the same vein, when we write \mathbf{U} we refer to $\mathbf{U}(P(Q), Q)$. We have

$$\frac{\partial Z}{\partial Q} = \pi \frac{\partial \mathbf{U}}{\partial Q} [1 + (q-1)\beta \mathbf{U}]^{\frac{1}{1-q}} = 2\pi Q [1 + (q-1)\beta \mathbf{U}]^{\frac{1}{1-q}}, \quad (6.3)$$

and

$$\begin{aligned} \frac{\partial \langle \mathbf{U} \rangle}{\partial Q} = & \frac{(2-q)}{(3-2q)(q-1)} \cdot \frac{2Q}{\left\{1 - [1 + (q-1)\beta \mathbf{U}]^{\frac{2-q}{1-q}}\right\}} \left\{ (3-2q) [1 + (q-1)\beta \mathbf{U}]^{\frac{2-q}{1-q}} \right. \\ & \left. - (2-q) [1 + (q-1)\beta \mathbf{U}]^{\frac{1}{1-q}} \frac{\{1 - [1 + (q-1)\beta \mathbf{U}]^{\frac{3-2q}{1-q}}\}}{\{1 - [1 + (q-1)\beta \mathbf{U}]^{\frac{2-q}{1-q}}\}} \right\}. \end{aligned} \quad (6.4)$$

Rearranging terms and adding and subtracting one we can obtain

$$\begin{aligned} \frac{\partial \langle \mathbf{u} \rangle}{\partial Q} &= \frac{2Q(2-q)}{(q-1)} \cdot \frac{[1 + (q-1)\beta \mathbf{u}]^{\frac{1}{1-q}}}{\left\{1 - [1 + (q-1)\beta \mathbf{u}]^{\frac{2-q}{1-q}}\right\}} \\ &\quad \left\{ [1 + (q-1)\beta \mathbf{u}] - 1 + 1 - \frac{(2-q)}{(3-2q)} \frac{\{1 - [1 + (q-1)\beta \mathbf{u}]^{\frac{3-2q}{1-q}}\}}{\{1 - [1 + (q-1)\beta \mathbf{u}]^{\frac{2-q}{1-q}}\}} \right\}, \quad (6.5) \end{aligned}$$

while (6.5) can be written as

$$\frac{\partial \langle \mathbf{u} \rangle}{\partial Q} = 2Q(2-q) \frac{[1 + (q-1)\beta \mathbf{u}]^{\frac{1}{1-q}}}{\left\{1 - [1 + (q-1)\beta \mathbf{u}]^{\frac{2-q}{1-q}}\right\}} \{\beta \mathbf{u} - \beta \langle \mathbf{u} \rangle\}. \quad (6.6)$$

Thus, we have for F_e the expression

$$\begin{aligned} F_e &= \left(\frac{\pi}{\beta}\right)^{q-1} 2Q(2-q)^{2-q} \frac{[1 + (q-1)\beta \mathbf{u}]^{\frac{1}{1-q}}}{\left(1 - [1 + (q-1)\beta \mathbf{u}]^{\frac{2-q}{1-q}}\right)^{2-q}} \left\{ \beta \mathbf{u} + \frac{1}{q-1} \right. \\ &\quad \left. - \frac{1}{(q-1)(3-2q)} \frac{(2-q)^2 \{1 - [1 + (q-1)\beta \mathbf{u}]^{\frac{3-2q}{1-q}}\}}{\{1 - [1 + (q-1)\beta \mathbf{u}]^{\frac{2-q}{1-q}}\}} \right\}. \quad (6.7) \end{aligned}$$

If $q \rightarrow 1$,

$$\frac{\partial Z}{\partial Q} \rightarrow 2\pi Q e^{-\beta \mathbf{u}}, \quad (6.8)$$

$$\frac{\partial \langle \mathbf{u} \rangle}{\partial Q} \rightarrow \frac{2Q e^{-\beta \mathbf{u}}}{1 - e^{-\beta \mathbf{u}}} \left[\frac{\beta \mathbf{u}}{1 - e^{-\beta \mathbf{u}}} - 1 \right], \quad (6.9)$$

and

$$F_e \rightarrow 2Q\beta \mathbf{u} \frac{e^{-\beta \mathbf{u}}}{(1 - e^{-\beta \mathbf{u}})^2}. \quad (6.10)$$

For q close to unity, a first order approximation yields

$$Z \approx \pi \mathbf{u}, \quad (6.11)$$

$$\frac{\partial Z}{\partial Q} \approx 2\pi Q(1 - \beta \mathbf{u}) \approx 2\pi Q, \quad (6.12)$$

$$\langle \mathbf{U} \rangle \approx 0, \quad (6.13)$$

$$\frac{\partial \langle \mathbf{U} \rangle}{\partial Q} \approx 2Q(1 - \beta \mathbf{U}) \approx 2Q, \quad (6.14)$$

and

$$F_e \approx \frac{\pi}{\beta} (\pi \mathbf{U})^{q-2} 2Q + (\pi \mathbf{U})^{q-1} 2Q \approx (\pi \mathbf{U})^{q-1} 2Q \left(\frac{1}{\beta \mathbf{U}} + 1 \right). \quad (6.15)$$

Finally, the entropic force simplifies to

$$F_e \approx (\pi \mathbf{U})^{q-1} \frac{2Q}{\beta \mathbf{U}}, \quad (6.16)$$

and for $q \rightarrow 1$,

$$F_e \approx \frac{2Q}{\beta \mathbf{U}}, \quad (6.17)$$

In agreement with [24].

7 Entropic Force on arbitrary phase space curves

More generally, for $\mathbf{U} = \mathbf{P}^2 + \mathbf{Q}^2$ one has

$$F_e = \left(\frac{\pi}{\beta} \right)^{q-1} 2Q(2-q)^{2-q} \frac{[1 + (q-1)\beta(\mathbf{P}^2 + \mathbf{Q}^2)]^{\frac{1}{1-q}}}{\left(1 - [1 + (q-1)\beta(\mathbf{P}^2 + \mathbf{Q}^2)]^{\frac{2-q}{1-q}}\right)^{2-q}} \left\{ \beta(\mathbf{P}^2 + \mathbf{Q}^2) + \frac{1}{q-1} - \frac{1}{(q-1)(3-2q)} \frac{\{1 - [1 + (q-1)\beta(\mathbf{P}^2 + \mathbf{Q}^2)]^{\frac{3-2q}{1-q}}\}}{\{1 - [1 + (q-1)\beta(\mathbf{P}^2 + \mathbf{Q}^2)]^{\frac{2-q}{1-q}}\}} \right\} \quad (7.1)$$

We see that the entropic force is confined to just a small region of phase space. This confinement effect i) grows with q and ii) leads to asymptotic freedom (zero force) outside such a region. The peak of the entropic force becomes more pronounced as q increases. Also, as the kinetic energy, associated to the momentum, grows, so does confinement.

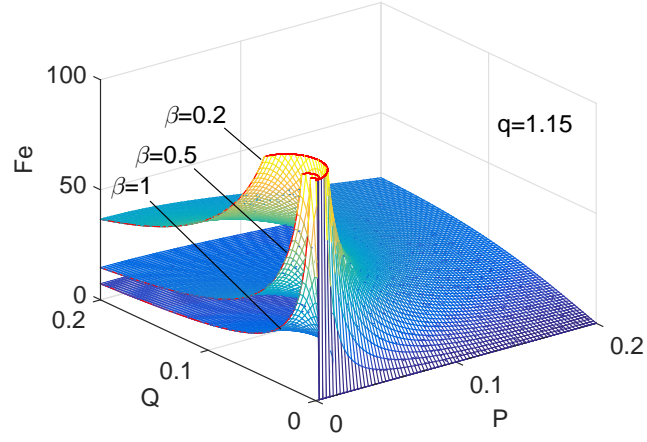


Figure 1: Entropic force. Note the the entropic force grows with the temperature, β as one should expect.

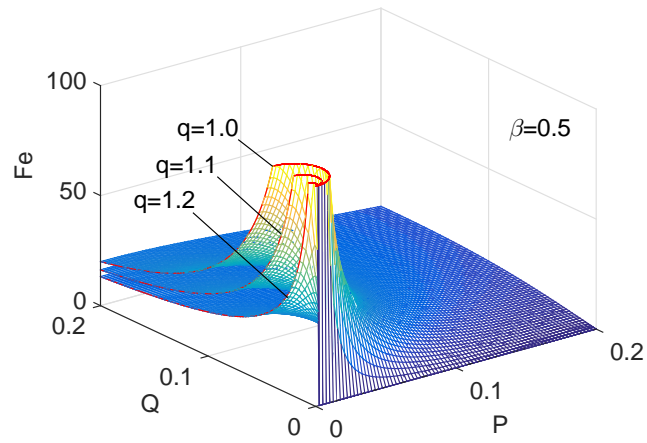


Figure 2: Entropic force. Note that the entropic force diminishes as q grows, which is a new result.

8 HO well

Here the particle also feels an harmonic force

$$F_{\text{HO}} = -\frac{1}{2} \frac{\partial \langle Q^2 \rangle}{\partial Q} = -\frac{1}{4} \frac{\partial \langle u \rangle}{\partial Q}. \quad (8.1)$$

In our case this becomes

$$F_{\text{HO}} = -\frac{Q}{2} (2 - q) \frac{[1 + (q - 1)\beta u]^{\frac{1}{1-q}}}{\left\{1 - [1 + (q - 1)\beta u]^{\frac{2-q}{1-q}}\right\}} \{\beta u - \beta \langle u \rangle\}. \quad (8.2)$$

Note that, for $q \rightarrow 1$,

$$F_{\text{HO}} \rightarrow -\frac{Q}{2} \frac{e^{-\beta u}}{(1 - e^{-\beta u})} \left(\frac{\beta u}{1 - e^{-\beta u}} - 1 \right), \quad (8.3)$$

or

$$F_{\text{HO}} \rightarrow \frac{Q}{2} \frac{e^{-\beta u}}{(1 - e^{-\beta u})^2} (1 - \beta u - e^{-\beta u}). \quad (8.4)$$

The total force that the particle feels is

$$F_{\text{T}} = F_{\text{e}} + F_{\text{HO}} = Z^{q-2} \frac{\partial Z}{\partial Q} \left(\frac{1}{\beta} + (q - 1) \langle u \rangle \right) + \frac{\partial \langle u \rangle}{\partial Q} \left(Z^{q-1} - \frac{1}{4} \right), \quad (8.5)$$

and then:

$$\begin{aligned}
F_T = & \left(\frac{\pi}{\beta}\right)^{q-1} 2Q(2-q)^{2-q} \frac{[1 + (q-1)\beta(P^2 + Q^2)]^{\frac{1}{1-q}}}{\left(1 - [1 + (q-1)\beta(P^2 + Q^2)]^{\frac{2-q}{1-q}}\right)^{2-q}} \\
& \left\{ \beta(P^2 + Q^2) + \frac{1}{q-1} \right. \\
& \left. - \frac{1}{(q-1)(3-2q)} \frac{(2-q)^2 \{1 - [1 + (q-1)\beta(P^2 + Q^2)]^{\frac{3-2q}{1-q}}\}}{\{1 - [1 + (q-1)\beta(P^2 + Q^2)]^{\frac{2-q}{1-q}}\}} \right\} - \\
& \frac{Q}{2}(2-q) \frac{[1 + (q-1)\beta(P^2 + Q^2)]^{\frac{1}{1-q}}}{\left\{1 - [1 + (q-1)\beta(P^2 + Q^2)]^{\frac{2-q}{1-q}}\right\}} \\
& \left[\beta(P^2 + Q^2) + \frac{1}{(q-1)} \left(1 - \frac{(2-q)}{(3-2q)} \frac{\{1 - [1 + (q-1)\beta(P^2 + Q^2)]^{\frac{3-2q}{1-q}}\}}{\{1 - [1 + (q-1)\beta(P^2 + Q^2)]^{\frac{2-q}{1-q}}\}} \right) \right] \\
& \tag{8.6}
\end{aligned}$$

For $q \rightarrow 1$,

$$F_T \rightarrow \frac{Q}{2} \frac{e^{-\beta U}}{(1 - e^{-\beta U})^2} (1 + 3\beta U - e^{-\beta U}). \tag{8.7}$$

Notice that, as above, the total force is confined to just a small region of phase space. Such confinement effect i) grows with q and ii) leads to asymptotic freedom (zero total force) outside such a region. Once again, the peak of the total force becomes more pronounced as q increases. Also, as total grows, so does confinement.

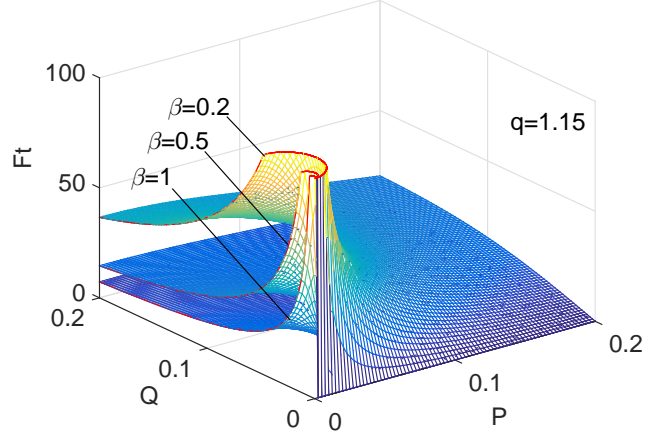


Figure 3: Total force. It grows as the temperature increases.

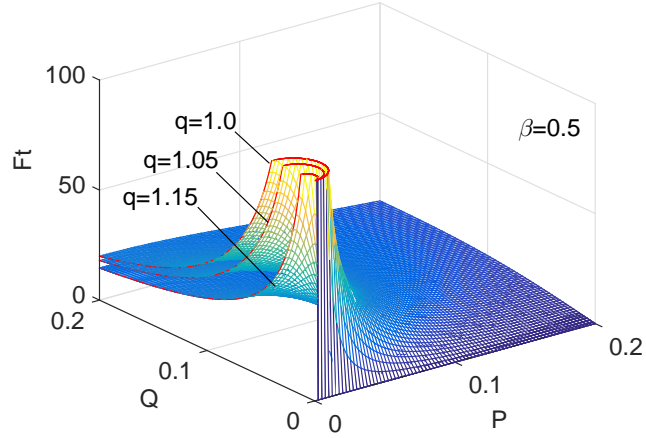


Figure 4: Total force. It diminishes as q grows.

9 Conclusions

In this paper we have been dealing with q -entropic-force effects. Although we discussed arbitrary phase space curves Γ , in general our effects do not depend upon the specific Γ . The q -statistical mechanics-along-curves no-

tion sounds reasonable because a q -equipartition theorem holds. We discovered that the q -entropic force diverges at specific small areas of phase space (hard-core effect), vanishing outside (confinement plus asymptotic freedom). Adding a harmonic well does not modify things. Also, the entropic force grows with q , an interesting result, and also with temperature, which makes a lot of sense. Finally, for curves, and because of q -equipartition, we discover a new upper bound for q , namely, $q < 3/2$.

References

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